Solution Sheet 2

1. If a function is a function known to be holomorphic in a certain domain it will only be stated without a proof.

   - $f(x+iy) = e^{-x}e^{-iy}$ implies that $f(z) = e^{-z}$ this function is entire, i.e. holomorphic in all of $\mathbb{C}$.
   - In this case $u(x,y) = x^2$ and $v(x,y) = y^2$. Hence $\dfrac{\partial u}{\partial x} = 2x$, $\dfrac{\partial v}{\partial y} = 2y$ and $\dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x} = 0$. The second Cauchy-Riemann equation holds automatically, as for the first we get $x = y$, hence the equations holds precisely on the diagonal in the plane.
   - Plug those formulae into the definition of the function to get:

     $$f(z) = \frac{(e^{ix}+e^{-ix})(e^{y}+e^{-y})}{2} - \frac{(e^{ix}-e^{-ix})(e^{y}-e^{-y})}{2} =$$
     $$= \frac{e^{ix+y}+e^{ix-y}+e^{-ix+y}+e^{-ix-y}-e^{ix+y}+e^{ix-y}+e^{-ix+y}-e^{-ix-y}}{2} =$$
     $$= e^{ix-y}+e^{-ix+y} = e^{iz}+e^{-iz}.$$

     This is an entire function.
   - We write out: $f(z) = (x+iy)^2 - (x-iy)^2 = x^2 + 2ixy - y^2 - x^2 + 2ixy + y^2 = 4ixy$. Hence $u(x,y) = 0$ and $v(x,y) = 4xy$. The Cauchy-Riemann equations then yield $4iy = 0$ amd $4ix = 0$. Hence they hold if and only if $x = y = 0$.

2. For $(x,y) \neq (0,0,)$ we have that $u(x,y) = \frac{x^2y}{x^2+y^2}$ and $v(x,y) = \frac{xy^2}{x^2+y^2}$. Let us compute the partial derivatives at $(0,0)$.

   $$\frac{\partial u}{\partial x}(0,0) = \lim_{h\to 0}\frac{u(h,0)-u(0,0)}{h} = 0,$$
   $$\frac{\partial u}{\partial y}(0,0) = \lim_{h\to 0}\frac{u(0,h)-u(0,0)}{h} = 0.$$

   Similarly $\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial y} = 0$. Hence the Cauchy-Riemann equations hold at 0. To show that f is not differentiable at 0 consider the sequence $z_n = \frac{1+i}{n}$, then:

   $$f(z_n) = \frac{\frac{1}{n^2}\frac{1+i}{n}}{\frac{2}{n^2}} = \frac{1+i}{2n}.$$

Hence:

$$\lim n \to \infty \frac{f(z_n) - f(0)}{z_n} = \frac{n}{1+i}\frac{1+i}{2n} = \frac{1}{2}.$$

Therefore the limit $\lim_{h\to 0} \frac{f(h)-f(0)}{h}$ doesn't exist.

3. Let $f(z) = u(x,y) + iv(x,y)$.

   - Since f is always real $v = 0$. Now apply the Cauchy-Riemann equations to get that $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$. Hence f is constant.
   - Write $f(z) = u(x,y)+iv(x,y)$, hence $\bar{f}(z) = u(x,y)-iv(x,y)$. For every $z = x+iy \in D$ the function f satisfies the Cauchy-Riemann equations, hence $\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y)$ and $\frac{\partial u}{\partial y}(x,y) = -\frac{\partial v}{\partial x}(x,y)$. By assumption $\bar{f}$ is also holomorphic, hence its real and imaginary parts satisfy the Cauchy-Riemann equations as well. Hence $\frac{\partial u}{\partial x}(x,y) = -\frac{\partial v}{\partial y}(x,y)$ and $\frac{\partial u}{\partial y}(x,y) = \frac{\partial v}{\partial x}(x,y)$. Summing the respective equations we get that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$ and hence also $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = 0$. Hence f is constant.
   - Since $|f|$ is constant we can write $u^2 + v^2 = C$. If $C = 0$ then clearly $u = v = 0$ and hence $f = 0$ and in particular f is constant. So we assume that $C \neq 0$, hence f has no zeroes. Therefore $g(z) = \frac{C}{f}$ is also holomorphic in D. On the other hand $C = f\bar{f}$, hence $g = \bar{f}$. However by a previous exercise f is constant.
   - We have that $Au(x,y)+Bv(x,y)+C = 0$, for every $z = x+iy \in D$. Taking partial derivatives we get:

$$A\frac{\partial u}{\partial x} + B\frac{\partial v}{\partial x} = 0,$$
$$A\frac{\partial u}{\partial y} + B\frac{\partial v}{\partial y} = 0.$$

Since f is holomorphic we get that:

$$A\frac{\partial u}{\partial x} - B\frac{\partial u}{\partial y} = 0,$$
$$A\frac{\partial u}{\partial y} + B\frac{\partial u}{\partial x} = 0.$$

Now if either A or B is 0 then the other is not and hence $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$. Using the Cauchy-Riemann equations once more we deduce that f is constant. Assume now that both $A \neq 0$ and $B \neq 0$. Then we multiply the first equation by A and the second by B and add:

$$(A^2 + B^2)\frac{\partial u}{\partial x} = 0.$$

Since $A^2 + B^2 > 0$ we get that $\frac{\partial u}{\partial x} = 0$. Using the above equations we get that $\frac{\partial u}{\partial y} = 0$ as well. Hence f is constant.

4. We will show that for every $z \in D$ the derivative exists using the definition. Write:

$$\lim_{h\to 0} \frac{g(z+h) - g(z)}{h} = \lim_{h\to 0} \frac{\overline{f(\overline{z+h})} - \overline{f(\bar{z})}}{h} = \lim_{h\to 0} \overline{\frac{f(\bar{z}+\bar{h}) - f(\bar{z})}{\bar{h}}} =$$
$$= \overline{\lim_{h\to 0} \frac{f(\bar{z}+\bar{h}) - f(\bar{z})}{\bar{h}}}.$$

The last limit exists since $\bar{z} \in D$ and f is holomorphic in D and since the adjoint is continuous $\bar{h} \to 0$.

5. We will use the fact that f is complex differentiable.

- Compute the derivative of f in two ways. Set $z = re^{i\theta}$ and calculate:

$$f'(z) = \lim_{t\to 0} \frac{f(re^{i\theta} + te^{i\theta}) - f(te^{i\theta})}{te^{i\theta}} =$$
$$= e^{-i\theta} \lim_{t\to 0} \frac{u(r+t,\theta) + iv(r+t,\theta) - u(r,\theta) - iv(r+t,\theta)}{t} =$$
$$= e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right).$$

Hence:

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = e^{i\theta} f'(z).$$

We can also compute the derivative as follows:

$$f'(z) = \lim_{t\to 0} \frac{f(re^{i(\theta+t)}) - f(re^{i\theta})}{re^{i(\theta+t)} - re^{i\theta}} =$$
$$= \frac{1}{r}\lim_{t\to 0} \frac{u(r,\theta+t) + iv(r,\theta+t) - u(r,\theta) - iv(r\theta)}{e^{i(\theta+t)} - e^{i\theta}} =$$
$$= \frac{1}{r}\lim_{t\to 0} \frac{u(r,\theta+t) + iv(r,\theta+t) - u(r,\theta) - iv(r\theta)}{t} \frac{t}{e^{i(\theta+t)} - e^{i\theta}} =$$
$$= \frac{1}{ire^{i\theta}}\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right).$$

Hence:

$$ire^{i\theta} f'(z) = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}.$$

Multiply the for the partial derivatives along r by ir and subtract the equation for the partial derivatives along θ, to get:

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} - ir\frac{\partial u}{\partial r} + r\frac{\partial v}{\partial r} = 0.$$

Now both the real and imaginary parts of this equation have to be 0.

- This is identical.

6. First note that g is defined on the set $\{z \in \mathbb{C} \mid |z| > R\}$. To see this note that if $z = re^{i\theta}$ then $\bar{z} = re^{-i\theta}$ and $\frac{R^2}{\bar{z}} = \frac{R^2}{r}e^{i\theta}$. For g to be well defined we need $\frac{R^2}{\bar{z}} \in D_R(0)$. Hence in particular $\frac{R^2}{r} < R$, which in turn implies that $R < r$. Now write $f(z) = u(r,\theta) + iv(r,\theta)$, by the above computation we get that $g(z) = \tilde{u}(r,\theta) + i\tilde{v}(r,\theta)$, where $\tilde{u}(r,\theta) = u(R^2/r,\theta)$ and $\tilde{v}(r,\theta) = -v(R^2/r,\theta)$. Now compute:

$$\frac{\partial \tilde{u}}{\partial r}(r,\theta) = \frac{\partial u}{\partial r}(R^2/r,\theta)(-\frac{R^2}{r^2}) = \frac{r}{R^2}\frac{\partial v}{\partial \theta}(R^2/r,\theta)(-\frac{R^2}{r^2}) =$$
$$= -\frac{1}{r}\frac{\partial v}{\partial \theta}(R^2/r,\theta) = \frac{1}{r}\frac{\partial \tilde{v}}{\partial \theta}(r,\theta).$$

The second computation is very similar.

7. We will use the formula for the radius of convergence.

   - Compute:

     $$\limsup_{n\to\infty} \sqrt[n]{|a|^{n^2}} = \limsup_{n\to\infty} |a|^n = \begin{cases} \infty, & |a| > 1, \\ 1, & |a| = 1, \\ 0, & |a| < 1. \end{cases}$$

     Hence the radius of convergence $R = \begin{cases} 0, & |a| > 1 \\ 1, & |a| = 1, \\ \infty, & |a| < 1 \end{cases}$.

   - In this case the coefficients are the sequence $a_m = \begin{cases} 0, & m \neq n!, \\ 1, & m = n! \end{cases}$.

     Hence $\limsup_{m\to\infty} \sqrt[m]{a_m} = 1$ and therefore the radius of convergence is 1.

     Another way to see it is by noting that this series is a subseries of $\sum_{n=0}^{\infty} z^n$ and hence its radius of convergence is at least 1. On the other hand clearly for $z = 1$ the series diverges, therefore the radius of convergence is exactly 1.

   - Clearly $\limsup_{n\to\infty} \sqrt[n]{n^{-n}} = 0$, hence the radius of convergence is ∞.

8. The following power series will be examples of the properties required (there are many others, if you've found a good example and can prove come show it to one of the TAs):

   - The series $\sum_{n=0}^{\infty} \frac{z^n}{n^2}$ have the unit disc as their domain of convergence. Indeed use the formula for radius of convergence:

     $$\limsup_{n\to\infty} \frac{1}{\sqrt[n]{n^2}} = 1.$$

     We can write, for ever z, such that $|z| = 1$:

     $$\left| \sum_{n=0}^{\infty} \frac{z^n}{n^2} \right| \leq \sum_{n=0}^{\infty} \frac{1}{n^2} < \infty.$$

- The series $\sum_{n=0}^{\infty} nz^n$ converge in the unit disc as well. One can see it either by utilizing the radius of convergence formula again or by simply noting that $\sum_{n=0}^{\infty} nz^n = z\left(\sum_{n=0}^{\infty} z^n\right)'$ and recalling that the derivative will have the same radius of convergence. Now if $|z| = 1$ then $z = e^{i\theta}$ for some real θ. Plugging it into the series we get $\sum_{n=0}^{\infty} ne^{i\theta}$, note that the general term does not tend to 0, hence the series diverges.

9. The series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ will serve as an example (this are constant functions). If $\sum_{n=0}^{\infty} M_n$ dominates this series then $M_n > \frac{1}{n}$ therefore $\sum_{n=0}^{\infty} M_n$ diverges.

10. If $|z| < 1-\delta$, then clearly $\sum_{n=0}^{\infty} z^n$ is dominated by the series $\sum_{n=0}^{\infty}(1-\delta)^n = \frac{1}{\delta}$. Therefore by the Weierstrass M-test the series converges uniformly.

    Now if $|z| < 1$, then we need to show that the remainder of the series does not converge uniformly to 0. Fix an integer N. The remainder $\sum_{n=N}^{\infty} z^n = \frac{z^N}{1-z}$. Take $z = 1 - \frac{1}{N}$. Then the remainder is $N(1-\frac{1}{N})^N$. For N sufficiently large this number is close to $\frac{N}{e}$, hence it does not go to 0. It implies that we don't have uniform convergence (independent of z).

11. - Write $p(z) = A\Pi_{k=1}^{n}(z - z_k)$. Then $p'(z) = A(\sum_{j=1}^{n} \Pi_{\substack{k=1\\k\neq j}}^{n}(z - z_k)$. Hence the logarithmic derivative is:

    $$\frac{p'(z)}{p(z)} = \sum_{j=1}^{n} \frac{1}{z - z_j}.$$

    - Compute:

    $$\mathrm{Im}(\frac{v-u}{b}) = \mathrm{Im}(\frac{v-a}{b}) - \mathrm{Im}(\frac{u-a}{b}) > 0.$$

    - Assume that $z_0 \notin H$ is a root of the derivative of p. In particular z_0 is not a root of p, since all of its roots are in H. Hence z_0 is a zero of the logarithmic derivative of p, i.e.:

    $$\sum_{j=1}^{n} \frac{1}{z_0 - z_j} = 0.$$

Note that since the sign of reciprocals are opposite, applying the second part of the exercise yields:

$$\mathrm{Im}(\frac{b}{z_0 - z_j} < 0.$$

The above inequality holds for every $1 \leq j \leq n$. Now multiply the logarithmic derivative of p at z_0 by b and take the imaginary part to get:

$$\mathrm{Im}(\frac{bp'(z_0)}{p(z_0)}) = \sum_{j=1}^{n} \mathrm{Im}(\frac{b}{z_0 - z_j}) < 0.$$

However it contradicts the fact that z_0 is a zero of the logarithmic derivative of p.

12. Define an auxiliary function $g(z) = \frac{1}{f(\frac{1}{\bar{z}})}$. Note that on the unit circle $z\bar{z} = 1$, hence $g(z)/f(z) = 1$ on the unit circle (since if $z = e^{i\theta}$ $z^{-1} = \bar{z}$). This implies that on the unit circle $g(z) = f(z)$, in particular $f(z) - g(z) = 0$ on the unit circle. However $f(z) - g(z)$ is a rational function and hence has only a finite number of zeroes, Hence $f(z) = g(z)$ everywhere in $\mathbb{C}$. So if z is a zero of p then $\frac{1}{\bar{z}}$ is a zero of q and vice versa. Hence f has the following form:

$$f(z) = A\frac{\Pi_{k=1}^{m}(z - \alpha_k)}{\Pi_{k=1}^{m}(z - \frac{1}{\bar{\alpha_k}})}.$$

Where $A \in \mathbb{C}$ is a constant.

13. Similarly as in the previous exercise let $g(z) = \overline{f(\frac{1}{\bar{z}})}$. Then since f is real on the unit circle we get that $f(z) = g(z)$ on the unit circle and hence everywhere in $\mathbb{C}$. Therefore if z is a zero of p so is $\frac{1}{\bar{z}}$ and similarly for q.